COMPARISON OF THE PERIODIC SOLUTIONS OF QUASI-LINEAR SYSTEMS CONSTRUCTED BY THE METHOD OF POINCARÉ AND BY THE METHOD OF KRYLOV-BOGOLIUBOV

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Poincaré's small-parameter method and the Krylov-Bogoliubov asymptotic method are among the number of basic methods used for the study of nonlinear oscillations. Poincaré's method was developed in conformity with stationary (periodic) oscillations [1], although it may be extended to nonstationary oscillations as well (see, for example, [2]). The Krylov-Bogoliubov method may be used, first of all, for a study of nonstationary oscillations, but it is, of course, completely applicable to periodic oscillations as well [3].

It is sometimes asserted that these methods are different in principle. Thus, for example, Poincaré's method requires the convergence of series in a small parameter which represent periodic solutions. On the other hand, in the description of the Krylov-Bogoliubov method it is emphasized that the question of the convergence of small-parameter expansions does not arise at all and that in some cases these series are known to be divergent. It is pointed out that the expansions used serve only for the construction of asymptotic approximations of any desired degree of accuracy under the condition that the small parameter approaches zero.

In the present paper we consider the periodic solutions of quasilinear systems with one degree of freedom, and the calculations are shown only for self-contained systems. A comparison is made between the first few terms of expansions obtained by the two methods.

1. We consider the self-contained oscillatory system

$$x^{\cdot\cdot} + \omega^2 x = \mu f(x, x^{\cdot}) \qquad (\ )^{\cdot} = d/dt \tag{1.1}$$

Let the function $f(x,x^{\cdot})$ be a polynomial or an analytic function of two arguments in some domain of their variation, and let μ be a small parameter.

According to the Krylov-Bogoliubov method, we attempt to find a solution in the form of an expansion in the small parameter (the notations vary somewhat):

$$x = a\cos\psi + \mu u_1 (a, \psi) + \mu^2 u_2(a, \psi) + \ldots \tag{1.2}$$

in which the quantities a and ψ satisfy Equations

$$a^{\cdot} = \mu A_1(a) + \mu^2 A_2(a) + \ldots, \qquad \psi^{\cdot} = \omega + \mu B_1(a) + \mu^2 B_2(a) + \ldots \tag{1.3}$$

The functions $u_n(a,\psi)$ are periodic functions of ψ, with period 2π, which do not contain the first harmonics in ψ. No initial conditions are introduced.

For periodic oscillations we have

$$a^{\cdot} = 0, \qquad \psi^{\cdot} = \text{const}$$

Consequently, if we assume that $\psi = 0$ when $t = 0$, then

$$\psi = t[\omega + \mu B_1(a) + \mu^2 B_2(a) + \ldots] \tag{1.4}$$

We write a, the amplitude of the first harmonic, in the form of an expansion

$$a = a_0 + a_1\mu + a_2\mu^2 + \ldots \tag{1.5}$$

Then the solution of Equation (1.1) may be represented in the form

$$x(\psi) = x_0(\psi) + \mu x_1(\psi) + \mu^2 x_2(\psi) + \ldots \tag{1.6}$$

where the coefficients of this expansion will be periodic functions of ψ with period 2π. For the first three coefficients, we obtain Expressions (1.7)

$$x_0(\psi) = a_0 \cos\psi, \quad x_1(\psi) = a_1 \cos\psi + u_1(a_0, \psi), \quad x_2(\psi) = a_2\cos\psi + a_1\left(\frac{\partial u_1}{\partial a}\right)_{a=a_0} + u_2(a_0, \psi)$$

It should be noted that in [1] the nth approximation means the sum of the first n terms of the series (1.6). For example, as the third approximation we have

$$x^{(3)}(\psi) = x_0(\psi) + \mu x_1(\psi) + \mu^2 x_2(\psi) = (a_0 + a_1\mu + a_2\mu^2)\cos\psi + \mu u_1(a_0 + a_1\mu, \psi) + \mu^2 u_2(a_0, \psi)$$

where ψ is taken with the required accuracy in each term.

The quantities $A_n(a)$ and $B_n(a)$ are defined by Formulas (1.8)

$$A_n(a) = -\frac{1}{2\pi\omega}\int_0^{2\pi} f_{n-1}(a, \psi)\sin\psi\, d\psi, \qquad B_n(a) = -\frac{1}{2\pi\omega a}\int_0^{2\pi} f_{n-1}(a, \psi)\cos\psi\, d\psi$$

The functions $f_n(a,\psi)$ for $n = 0, 1, 2$ are of the following form:

$$f_0(a, \psi) = f(a\cos\psi, -\omega a \sin\psi)$$

$$f_1(a, \psi) = u_1 f_x' + \left(A_1\cos\psi - a B_1 \sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right) f_{x^{\cdot}}' + \left(aB_1^2 - A_1\frac{dA_1}{da}\right)\cos\psi + A_1\left(2B_1 + a\frac{dB_1}{da}\right)\sin\psi - 2\omega A_1 \frac{\partial^2 u_1}{\partial a\,\partial\psi} - 2\omega B_1\frac{\partial^2 u_1}{\partial\psi^2}$$

$$f_2(a, \psi) = \frac{1}{2}u_1^2 f_{xx}'' + u_1\left(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right) f_{xx^{\cdot}}'' + \frac{1}{2}\left(A_1\cos\psi - aB_1\sin\psi + \omega\frac{\partial u_1}{\partial\psi}\right)^2 f_{x^{\cdot}x^{\cdot}}'' + u_2 f_x' + \left(A_2\cos\psi - aB_2\sin\psi + A_1\frac{\partial u_1}{\partial a} + B_1\frac{\partial u_1}{\partial\psi} + \omega\frac{\partial u_2}{\partial\psi}\right) f_{x^{\cdot}}' + \left(2aB_1B_2 - A_1\frac{dA_2}{da} - A_2\frac{dA_1}{da}\right)\cos\psi + \left(2A_1B_2 + 2A_2B_1 + aA_1\frac{dB_2}{da} + aA_2\frac{dB_1}{da}\right)\sin\psi + \ldots$$

The omitted terms in the last formula contain the partial derivatives of the functions $u_1(a,\psi)$ and $u_2(a,\psi)$, with coefficients which depend only on a.

In the preceding formulas all the derivatives of the function $f(x,x^{\cdot})$ are calculated for $x = a\cos\psi$, $x^{\cdot} = -\omega a\sin\psi$. Moreover, in these formulas, as in the subsequent discussion, the quantity a is taken to mean the value of this quantity when $\mu = 0$, that is, $a = a_0$.

The functions $u_n(a,\psi)$ are solutions of the differential equation

$$\frac{\partial^2 u_n}{\partial\psi^2}+u_n=\frac{1}{\omega^2}\left[f_{n-1}(a,\psi)+2\omega A_n\sin\psi+2\omega a B_n\cos\psi\right] \tag{1.9}$$

Let us now consider the construction of periodic solutions of equation (1.1) by Poincaré's method. For self-contained systems the solutions are usually constructed with the initial consition $x^{\cdot}(0) = 0$. The coefficients of the series (1.6) are obtained in the following form [4]:

$$x_0(\psi)=A_0^*\cos\psi,\quad x_1(\psi)=A_1^*\cos\psi+C_1(\psi)-h_1A_0^*\psi\sin\psi$$

$$x_2(\psi)=A_2^*\cos\psi+C_2(\psi)+A_1^*\frac{\partial C_1(\psi)}{\partial A_0^*}+h_1\psi C_1'(\psi)-$$

$$-(h_2A_0^*+h_1A_1^*)\psi\sin\psi-\frac{1}{2}h_1^2A_0^*\psi^2\cos\psi \tag{1.10}$$

The functions $C_n(\psi)$ are defined by Formula

$$C_n(\psi)=\frac{1}{\omega^2}\int_0^{\psi}H_n(\psi_1)\sin(\psi-\psi_1)d\psi_1 \tag{1.11}$$

where, for $n = 1, 2, 3,$ we have

$$H_1(\psi)=f(A_0^*\cos\psi,\ -\omega A_0^*\sin\psi)$$

$$H_2(\psi)=f_x'C_1(\psi)+\omega f_{x^{\cdot}}'C_1'(\psi)$$

$$H_3(\psi)={}^1/_2 f_{xx}''C_1^2(\psi)+\omega f_{xx^{\cdot}}''C_1(\psi)C_1'(\psi)+{}^1/_2\omega^2 f_{x^{\cdot}x^{\cdot}}''C_1'^2(\psi)+$$

$$+f_x'C_2(\psi)+\omega f_{x^{\cdot}}'C_2'(\psi)$$

It should be noted that in constructing the series (1.6) by Poincaré's method, we use a transformation of the independent variable

$$\omega t=\psi(1+h_1\mu+h_2\mu^2+\ldots) \tag{1.12}$$

The coefficients h_1, h_2 and h_3 have the following values:

$$h_1=\frac{1}{2\pi}N_1,\qquad h_2=\frac{1}{2\pi}\left(A_1^*\frac{\partial N_1}{\partial A_0^*}+N_2\right) \tag{1.13}$$

$$h_3=\frac{1}{2\pi}\left(A_2^*\frac{\partial N_1}{\partial A_0^*}+\frac{1}{2}A_1^{*2}\frac{\partial^2 N_1}{\partial A_0^{*2}}+A_1^*\frac{\partial N_2}{\partial A_0^*}+N_3\right)$$

In these formulas the following notation is used:

$$N_1=\frac{1}{A_0^*}C_1'(2\pi),\quad N_2=\frac{1}{A_0^*}\left[C_2'(2\pi)+\frac{1}{\omega^2}N_1H_1(2\pi)\right]$$

$$N_3=\frac{1}{A_0^*}\left\{C_3'(2\pi)+\frac{1}{\omega^2}N_2H_1(2\pi)-N_1\left[C_2(2\pi)+\right.\right.$$

$$\left.\left.+\frac{1}{3A_0^*}C_1'^2(2\pi)-\frac{1}{2\omega^2A_0^*}H_1'(2\pi)C_1'(2\pi)-\frac{1}{\omega^2}H_2(2\pi)\right]\right\}$$

From Formula (1.12) we obtain

$$\psi=\omega t\left[1-h_1\mu+(h_1^2-h_2)\mu^2-(h_1^3-2h_1h_2+h_3)\mu^3+\ldots\right] \tag{1.14}$$

2. Comparing the coefficients for equal powers of μ in the right-hand parts of Formulas (1.4) and (1.14), we find

$$-\omega h_1=B_1(a),\qquad \omega(h_1^2-h_2)=a_1\frac{dB_1}{da}+B_2(a) \tag{2.1}$$

$$-\omega(h_1^3-2h_1h_2+h_3)=a_2\frac{dB_1}{da}+\frac{1}{2}a_1^2\frac{d^2B_1}{da^2}+a_1\frac{dB_2}{da}+B_3(a)$$

In order to determine the relationship between the functions $C_1(\psi)$ and $u_1(a,\psi)$, we integrate Equation (1.9) for $n = 1$. Taking Formula (1.11) into account, we obtain

$$C_1(\psi) = u_1(a,\psi) - u_1(a,0)\cos\psi - \left(\frac{\partial u_1}{\partial \psi}\right)_{\psi=0}\sin\psi -$$
$$- \frac{1}{\omega} A_1(a)(\sin\psi - \psi\cos\psi) - \frac{a}{\omega} B_1(a)\psi\sin\psi \qquad (2.2)$$

If we set $\psi = 2\pi$ in this equation, then

$$C_1(2\pi) = (2\pi/\omega) A_1(a) \qquad (2.3)$$

Next, we differentiate Equation (2.2) with respect to ψ and set $\psi = 2\pi$ in this equation. Then

$$C_1'(2\pi) = (-2\pi a/\omega) B_1(a) \qquad (2.4)$$

Formulas (2.3) and (2.4) can be obtained directly from (1.8).

In Poincaré's method the coefficient A_0^* is found from Equation

$$C_1(2\pi) = 0$$

Correspondingly, in the Krylov-Bogoliubov method the coefficient a_0 is determined from Equation

$$A_1(a_0) = 0$$

On the basis of Equation (2.3), we then have

$$A_0^* = a_0 \qquad (2.5)$$

Consequently, the first approximation of $x_0(\psi)$ is the same in the two methods. Taking Formulas (1.13) and (2.4) into account, we readily see that the first equation of (2.1) becomes an identity.

Hereafter we shall assume that A_0^*, and consequently a_0 as well, will be simple roots of the equations from which they are determined.

Calculating functions $A_2(a)$ and $B_2(a)$ by Formulas (1.8), after some transformations we obtain

$$A_2(a) = \frac{\omega}{2\pi}\left[C_2(2\pi) + \frac{\partial C_1}{\partial A_0^*} u_1(a,0) + \frac{1}{2A_0^*} C_1'^2(2\pi)\right]$$

$$B_2(a) = -\frac{\omega}{2\pi a}\left\{C_2'(2\pi) + \frac{\partial C_1'}{\partial A_0^*} u_1(a,0) - \frac{1}{2\pi A_0^*} C_1'^2(2\pi) -\right.$$
$$\left. - \frac{1}{A_0^*}\left[u_1(a,0) - \frac{1}{\omega^2} f_0(a,0)\right] C_1'(2\pi)\right\} \qquad (2.6)$$

In order to determine the coefficient A_1^* in Poincaré's method, we have Equation

$$A_1^* \frac{\partial C_1}{\partial A_0^*} + C_2(2\pi) + \frac{1}{2A_0^*} C_1'^2(2\pi) = 0$$

The analogous equation for a_1 in the Krylov-Bogoliubov method will be

$$a_1 \frac{dA_1}{da} + A_2(a) = 0$$

Taking the expressions for $A_1(a)$ and $A_2(a)$ into account, we find the relationship betwee A_1^* and a_1

$$A_1^* = a_1 + u_1(a_0, 0) \qquad (2.7)$$

We compare the coefficients of μ in the expansion (1.6) which are obtained by the two methods. For this purpose, in the second formula of (1.10) we substitute the expressions for A_1^*, $C_1(\psi)$, and h_1 from the appropriate formulas. We obtain

$$x_1(\psi) = a_1\cos\psi + u_1(a,\psi) - \left(\frac{\partial u_1}{\partial \psi}\right)_{\psi=0}\sin\psi$$

On the other hand, by the Krylov-Bogoliubov method we have Formula (1.7) for $x_1(\psi)$.

However, a comparison of the periodic solutions must be carried out for identical initial conditions. Let us assume that the periodic solution according to the Krylov-Bogoliubov method is also constructed with the initial condition $x^{\bullet}(0) = 0$. Then

$$\left(\frac{\partial u_1}{\partial \psi}\right)_{\psi=0} = 0 \tag{2.8}$$

Here the function $x_1(\psi)$ determined be the two methods will be identical.

It is easy to verify that the second equation of (2.1) becomes an identity if we replace all the quantities appearing in it by the known expressions for them.

Next, we integrate Equation (1.9) for $n = 2$. Taking Formula (1.11) into account, we find

$$C_2(\psi) = u_2(a, \psi) - u_2(a, 0)\cos\psi - \left(\frac{\partial u_2}{\partial \psi}\right)_{\psi=0}\sin\psi -$$
$$- \frac{\partial C_1(\psi)}{\partial A_0^*} u_1(a, \psi) + \frac{1}{\omega} B_1(a)\psi\frac{\partial u_1}{\partial \psi} - \frac{a}{2\omega^2} B_1^2(a)\psi^2\cos\psi +$$
$$+ \frac{1}{\omega} A_2(a)(\psi\cos\psi - \sin\psi) - \frac{a}{\omega} B_2(a)\psi\sin\psi \tag{2.9}$$

Let us now calculate the coefficients $A_3(a)$ and $B_3(a)$. After some complicated transformations we obtain

$$A_3(a) = \frac{\omega}{2\pi}\left[C_3(2\pi) + \frac{\partial C_2}{\partial A_0^*} u_1(a, 0) + \frac{1}{2}\frac{\partial^2 C_1}{\partial A_0^{*2}} u_1^2(a, 0) + \frac{\partial C_1}{\partial A_0^*} u_2(a, 0)\right] +$$
$$+ \frac{1}{\omega} B_1 A_2 - \frac{\pi}{\omega} A_2 \frac{dA_1}{da} - A_2 \frac{\partial u_1(a, 0)}{\partial a} + \frac{2\pi a}{\omega} B_1 B_2 - \frac{2\pi a}{\omega^2} B_1^3 +$$
$$+ \frac{\pi}{\omega} B_1^2 \left[u_1(a, 0) - \frac{1}{\omega^2} f_0(a, 0)\right]$$

$$B_3(a) = -\frac{\omega}{2\pi a}\left[C_3'(2\pi) + \frac{\partial C_2'}{\partial A_0^*} u_1(a, 0) + \frac{1}{2}\frac{\partial^2 C_1'}{\partial A_0^{*2}} u_1^2(a, 0) + \frac{\partial C_1'}{\partial A_0^*} u_2(a, 0)\right] -$$
$$- \frac{1}{a} B_1\left[u_2(a, 0) - \frac{1}{\omega^2} f_1(a, 0)\right] + \frac{1}{a}\left(\frac{1}{\omega} B_1^2 - B_2\right)\left[u_1(a, 0) - \frac{1}{\omega^2} f_0(a, 0)\right] +$$
$$+ \frac{1}{a} A_2 \left(\frac{\partial^2 u_1}{\partial a\, \partial \psi}\right)_{\psi=0} + \frac{1}{a\omega} A_2 \frac{dA_1}{da} - \frac{\pi}{\omega} A_2 \frac{dB_1}{da} - \frac{2\pi}{a\omega} A_2 B_1 + \frac{2}{\omega} B_1 B_2 +$$
$$+ \frac{2}{\omega^2} B_1^3\left(\frac{\pi^2}{3} - 1\right) + \frac{2}{a\omega} B_1^2 \left(\frac{\partial^2 u_1}{\partial \psi^2}\right)_{\psi=0} + \frac{\pi}{a\omega^3} B_1^2 \left(\frac{\partial f_0}{\partial \psi}\right)_{\psi=0} \tag{2.10}$$

The coefficient A_2^* in Poincaré's method is determined from Equation

$$A_2^* \frac{\partial C_1}{\partial A_0^*} + \frac{1}{2} A_1^{*2} \frac{\partial^2 C_1}{\partial A_0^{*2}} + A_1^*\left[\frac{\partial C_2}{\partial A_0^*} + \frac{1}{A_0^*}\frac{\partial C_1'}{\partial A_0^*} C_1'(2\pi) - \frac{1}{2A_0^{*2}} C_1'^2(2\pi)\right] +$$
$$+ C_3(2\pi) + \frac{1}{A_0^*} C_2'(2\pi) C_1'(2\pi) + \frac{1}{2\omega^2 A_0^{*2}} H_1(2\pi) C_1'^2(2\pi) = 0$$

The equation for the coefficient a_2 in the Krylov-Bogoliubov method will be

$$a_2 \frac{dA_1}{da} + \frac{1}{2} a^2 \frac{d^2 A_1}{da^2} + a_1 \frac{dA_2}{da} + A_3(a) = 0$$

Comparing the left-hand sides of these equations, we find

$$A_2^* = a_2 + a_1 \frac{\partial u_1(a_0, 0)}{\partial a_0} + u_2(a_0, 0) \tag{2.11}$$

In the third formula of (1.10) we replace all the quantities by the equivalent expressions. After some transformations we obtain

$$x_2(\psi) = a_2 \cos\psi + a_1 \frac{\partial u_1}{\partial a} + u_2(a, \psi) - \left[a_1 \frac{\partial^2 u_1}{\partial a\, \partial\psi} + \frac{\partial u_2}{\partial\psi}\right]_{\psi=0} \sin\psi$$

The function $x_2(\psi)$ in the Krylov-Bogoliubov method is determined from Formula (1.7).

If we consider the initial condition for this function, we find that

$$\left[a_1 \frac{\partial^2 u_1}{\partial a\, \partial\psi} + \frac{\partial u_2}{\partial\psi}\right]_{\psi=0} = 0 \qquad (2.12)$$

Consequently, the values of the function $x_2(\psi)$ that are determined by the two methods are exactly the same.

It is easy to verify that the third equation of (2.1), like the previous equations, becomes an identity.

Thus, all three approximations obtained by one method completely coincide with those obtained by the other. Obviously, any other approximations obtained by the two methods will also coincide.

For quasilinear non-self-contained systems a comparison between the two approximations has also been made in the case of principal resonance. The agreement was found to be complete, just as in the case of self-contained systems.

A similar comparison for systems with several degrees of freedom will obviously lead to analogous results. In particular, in Poincaré's method for single-frequency oscillations of quasilinear self-contained systems described by second-order equations, the problem of constructing periodic solutions may be reduced to a problem with one degree of freedom, with the additional calculation of a number of supplementary functions [5]. By the Krylov-Bogoliubov method, this problem is solved in a somewhat different manner [3]. The first approximations obtained by the two methods will coincide. A comparison of the second approximations has not been made, owing to the difficulty of the calculations. However, there is no need for this, since such a comparison might rather serve for verifying the correctness of applying one or the other method to the indicated problem but not for a comparison of the methods themselves.

The general conclusion to be drawn from the foregoing is this: Poincaré's small-parameter method and the Krylov-Bogoliubov asymptotic method are, in a certain sense, equivalent methods when applied to the problem of constructing the periodic oscillations of quasilinear systems. This means that two corresponding approximations calculated by the two methods will be identical.

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